ADVANCED MATHEMATICS

Final Exam - June 2013

Name: ____________________

NIU: ________________ Group: _____

Grade:

Instructions: The exam consists of six questions. You have two hours to give a reasoned answer to all the exercises. Write the quiz entirely in ink.

1 Determine for which values of the parameter $a \in \mathbb{R}$ the matrix A is diagonalizable.

$$A = \begin{pmatrix} 0 & 0 & a \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

2 Determine whether the following statement is true. If it is, provide a reasoned argument. If it is not, show a counterexample: *"If A is a matrix whose determinant is equal to zero, then A is NOT diagonalizable."*

3 Solve the system of difference equations

$$X_{t+1} = AX_t,$$

where A is the matrix of Exercise 1 when $a = 4$. Is this system globally asymptotically stable? Is there any initial condition X_0 such that the solution converges?

4 Solve the following differential equation:

$$x' = -\frac{2t + 3x}{3t + 2x},$$

where $x(1) = 1$.

5 Obtain the solutions of the following differential equation:

$$x''' - 3x'' + 3x' - x = t^2$$

6 Obtain the solution to the following system:

$$X' = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix} X + \begin{pmatrix} e^t \\ 1 \end{pmatrix},$$

	ADVANCED MATHEMATICS Final Exam - June 2013	SOLUTIONS

Instructions: The exam consists of six questions. You have two hours to give a reasoned answer to all the exercises. Write the quiz entirely in ink.

1 Determine for which values of the parameter $a \in \mathbb{R}$ the matrix A is diagonalizable.

$$A = \begin{pmatrix} 0 & 0 & a \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Solution. The eigenvalues of A are $\sigma(A) = \{1, +\sqrt{a}, -\sqrt{a}\}$. If $a \in \mathbb{R}_-$ then the characteristic polynomial has complex roots and A is not diagonalizable. If $a \in \mathbb{R}_+ \setminus \{0, 1\}$ then the three roots are real and different each other and A is diagonalizable. We study the other two cases:

(a) If $a = 0$. Then $\sigma(A) = \{1, 0\}$ with $m(1) = 1$ and $m(0) = 2$. On the other hand, $\dim S(0) = 3 - \text{rank}(A - 0 \cdot I_3) = 3 - 2 = 1$. Since the dimension of the eigenspace and the multiplicity do not coincide, the matrix A is not diagonalizable.

(b) If $a = 1$. Then $\sigma(A) = \{1, -1\}$ with $m(1) = 2$ and $m(-1) = 1$. We compute the dimensions of the eigenspaces $\dim S(1) = 3 - \text{rank}(A - 1 \cdot I_3) = 3 - 2 = 1$ and $\dim S(-1) = 3 - \text{rank}(A + 1 \cdot I_3) = 3 - 2 = 1$. Since $\dim S(1) \neq m(1)$, the matrix A is not diagonalizable.

2 Determine whether the following statement is true. If it is, provide a reasoned argument. If it is not, show a counterexample: *"If A is a matrix whose determinant is equal to zero, then A is NOT diagonalizable."*

Solution. The statement is false. The following matrix (the null matrix) has a determinant equal to zero and it is diagonalizable (indeed, it is diagonal).

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

3 Solve the system of difference equations

$$X_{t+1} = AX_t,$$

where A is the matrix of Exercise 1 when $a = 4$. Is this system globally asymptotically stable? Is there any initial condition X_0 such that the solution converges?

Solution. When $a = 4$, $\sigma(A) = \{1, 2, -2\}$, and $S(1) =< (0, 1, 0) >$, $S(2) =< (2, 3, 1) >$ and $S(-2) =< (-6, 1, 3) >$. The system is homogeneous, and therefore

$$X_t = A_0 1^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + A_1 2^t \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + A_2 (-2)^t \begin{pmatrix} -6 \\ 1 \\ 3 \end{pmatrix}$$

Since not all the eigenvalues are smaller than 1, the system is not globally asymptotically stable. On the other hand, the solution will converge for an initial condition X_0 only if $A_1 = A_2 = 0$

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = X_0 = A_0 1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} -6 \\ 1 \\ 3 \end{pmatrix} \quad \Leftrightarrow \quad \alpha = \gamma = 0.$$

For $X_0 = (0, 1, 0)$, for example, the solution converges.

4 Solve the following differential equation:

$$x' = -\frac{2t + 3x}{3t + 2x},$$

where $x(1) = 1$.

Solution. It is an exact equation whose canonical form is $(2t + 3x)dt + (2x + 3t)dx = 0$ In this case, $P(t, x) = 2t + 3x$ and $Q(t, x) = 2x + 3t$. We check the condition to be exact:

$$\frac{\partial P}{\partial x} = 3 = \frac{\partial Q}{\partial t}.$$

- Let $F(t, x)$ be the solution of the equation we are looking for.
- Impose that
$$\frac{\partial F(t, x)}{\partial t} = P(t, x) \Rightarrow \frac{\partial F(t, x)}{\partial t} = 2t + 3x,$$
by integrating both sides with respect to t, we can isolate $F(t, x)$.
$$F(t, x) = t^2 + h(x).$$
- Now, impose that
$$\frac{\partial F(t, x)}{\partial x} = Q(t, x) \;\Leftrightarrow\; h'(x) = 2x + 3t.$$
- To obtain $h(x)$, simply integrate.
$$h(x) = x^2 + 3tx.$$
- Substitute $h(x)$ in the expression of Step 2, and then:
$$F(t, x) = t^2 + x^2 + 3tx.$$
- The solution to the exact equation is given in implicit form:
$$t^2 + x^2 + 3tx = C$$
Now, we impose the initial condition $x(1) = 1$ to obtain that $C = 5$. Therefore, the solution is $t^2 + x^2 - 3tx = 5$.

5 Obtain the solutions of the following differential equation:

$$x''' - 3x'' + 3x' - x = t^2$$

Solution. The roots of the characteristic polynomial are $r = 1$ with $m(1) = 3$. Then,

$$x^h(t) = A_0e^t + A_1te^t + A_2t^2e^t$$

Since $b(t) = t^2$ is a polynomial of degree 2, we propose $x(t) = C_0 + C_1t + C_2t^2$. Taking derivatives and substituting we get that $C_0 = -12$, $C_1 = -6$, and $C_2 = -1$. And the,

$$x^p(t) = -12 - 6t - t^2$$

Finally

$$x(t) = A_0e^t + A_1te^t + A_2t^2e^t - 12 - 6t - t^2$$

6 Obtain the solution to the following system:

$$X' = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix} X + \begin{pmatrix} e^t \\ 1 \end{pmatrix},$$

Solution. The matrix A of the system is already diagonal, hence, D is the A and $P = P^{-1} = I_2$, which implies that the solution to the associated homogeneous system is:

$$X^h(t) = K_0e^{-3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + K_1e^t\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In order to obtain a particular solution, we solve the system

$$K_0' e^{-3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + K_1' e^{t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{t} \\ 1 \end{pmatrix} \quad \equiv \quad K_0' = e^{4t} \text{ and } K_1' = e^{-t} \quad \equiv \quad K_0 = \frac{1}{4} e^{4t} \text{ and } K_1 = -e^{-t}$$

Then, a particular solution is

$$X^p(t) = \frac{1}{4} e^{4t} e^{-3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + -e^{-t} e^{t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} e^{t} \\ -1 \end{pmatrix}$$

Finally,

$$X(t) = K_0 e^{-3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + K_1 e^{t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{4} e^{t} \\ -1 \end{pmatrix}$$